Assouad dimension and Fractal Geometry - Part 2

Iterated function systems (IFS)

$\{S_i\}_{i \in \mathcal{I}}$ $\quad S_i : [0,1]^d \to [0,1]^d$

$$|S_i(x) - S_i(y)| \leq c_i |x-y|$$

for some $c_i \in (0,1)$, $\forall x, y \in [0,1]^d$.

$|\mathcal{I}| < \infty$. (see Mauldin-Urbanski PLMS 1996 for infinite IFS)
and Banaji-Fraser arXiv: 2207.11611

Hutchinson: there exists a unique non-empty compact $F \subseteq [0,1]^d$ such that

$$F = \bigcup_{i \in \mathcal{I}} S_i(F).$$

F often "fractal"

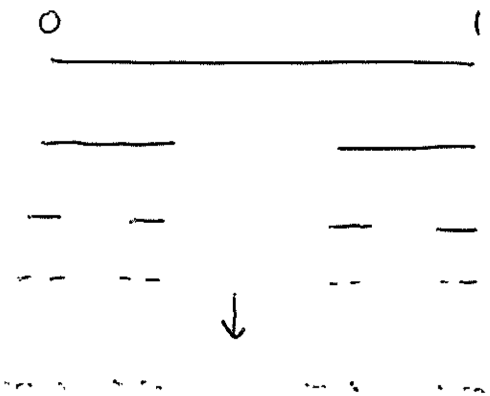
e.g. If $|S_i(x) - S_i(y)| = c_i |x-y|$

then S_i is a <u>similarity</u>.

If all S_i are similarities, the F is <u>self-similar set</u>.

For example: middle 3rd Cantor set is attractor of IFS

$$\left\{ x \mapsto x/3 ,\ x \mapsto \frac{x}{3} + \frac{2}{3} \right\}$$



Sierpinski triangle

Sierpinski carpet / sponge.

Question: what are the dimension of F?

<u>Theorem</u>: Suppose for $i \neq j \in \mathcal{I}$ $S_i((0,1)^d) \cap S_j((0,1)^d) = \emptyset$.

open set condition OSC

then $\dim_H F = \dim_B F = \dim_A F = s$

where

$$\sum_{i \in \mathcal{I}} c_i^s = 1.$$

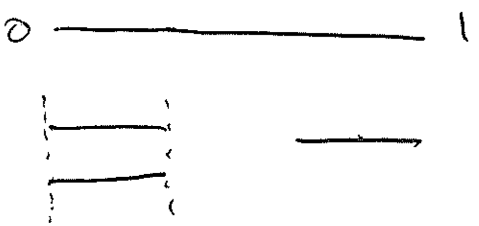
(Hutchinson–Moran formula)

Big question: when does s <u>not</u> give the dimension?

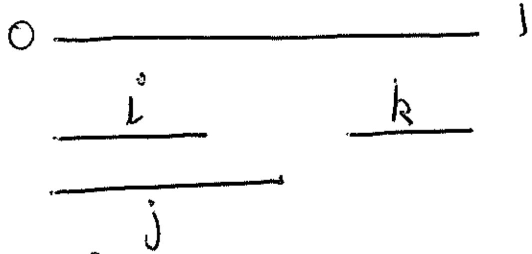
clearly s does not give dimension when there are exact overlaps (e.g. $\{\underline{x \mapsto x/3}, \underline{x \mapsto x/3}, x \mapsto x/3 + 2/3\}$)

Conjecture: If there are no exact overlaps (that is, $\mathrm{Semi}\langle S_i : i \in \mathcal{I} \rangle$ is free) then $\dim_H F = \min\{s, d\}$. (open).

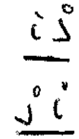
exact overlaps at level 1



exact overlaps at level 2



level 1



$\underbrace{ij = ji}_{\text{relation}}$

level 2

$\Rightarrow \mathrm{Semi}\langle S_i, S_j, S_k \rangle$ is not free.

Many recent breakthroughs:

Hochman (2014 Annals)

Rapaport (2022 Ann ENS.)

Theorem (Falconer):

$$\dim_H F = \dim_B F$$

for <u>all</u> self-similar sets F.

Question (Olsen 2011): Is it true that

$\dim_A F = \dim_H F$ for all self-similar sets?

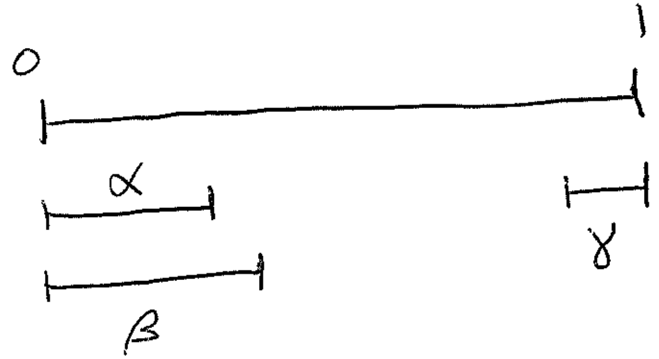
Answer: No. (Fraser TAMS 2014).

Consider IFS: $\{ x \mapsto \alpha x,\ x \mapsto \beta x,\ x \mapsto \gamma x + (1-\gamma) \}$

where $\alpha, \beta, \gamma \in (0,1)$ are very small: $\underbrace{\alpha + \beta + \gamma < 1}_{\Rightarrow\ s < 1}$.

and $\dfrac{\log \alpha}{\log \beta} \notin \mathbb{Q}$.

claim: $\dim_A F = 1$, $\underbrace{\dim_H F \leq s < 1}_{\text{clear.}}$



I will prove that $[0,1]$ is a weak tangent to F.

Consider $T_k : \mathbb{R} \to \mathbb{R}$, $T_k(x) = \beta^{-k} x$

e.g. $T_k([0, \beta^k]) = [0,1]$

Want: $T_k(F) \cap [0,1] \longrightarrow_{d_{\mathcal{H}}} [0,1]$.

$$T_k(F) \cap [0,1] \supseteq \{ \alpha^m \beta^n : m \geq 0, \ n \geq -k \} \cap [0,1]$$

Any limit of $T_k(F) \cap [0,1]$ contains $\overline{\{\alpha^m \beta^n : m \geq 0, n \in \mathbb{Z}\} \cap [0,1]}$

It remains to show that $\overline{\{\alpha^m\beta^n : m\in\mathbb{N}, n\in\mathbb{Z}\}}\cap[0,1]$ is in fact $[0,1]$.

$$\log(\alpha^m\beta^n) = m\log\alpha + n\log\beta$$
$$= n\log\alpha\underbrace{\left(\frac{m}{n}+\frac{\log\beta}{\log\alpha}\right)}_{\text{I can make this small!}}$$

Dirichlet's theorem: For all $\eta\notin\mathbb{Q}$ there exist infinitely many $m\in\mathbb{N}$, $n\in\mathbb{Z}$ such that $\gcd(m,n)=1$.

and $$\left|\eta-\frac{m}{n}\right|\leq\frac{1}{n^2}$$

therefore $\log(\alpha^m \beta^n)$ can be made arbitrarily small and this shows $\{\log(\alpha^m\beta^n) : m \in \mathbb{N}, n \in \mathbb{Z}\}$ is dense in $(-\infty, 0)$.

This completes the proof, and we have shown $\dim_A F = 1$.

Theorem (Fraser, Henderson, Olson, Robinson Adv. 2015)

Either ① $\dim_H F = \dim_A F$ (weak separation condition)

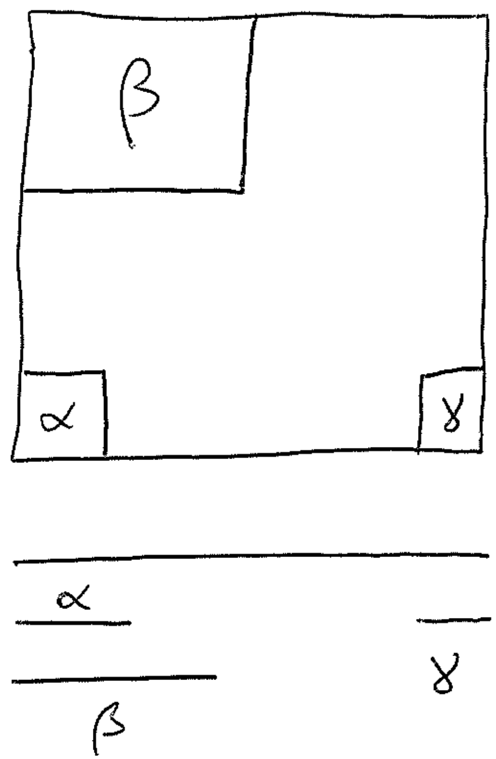
or ② $\dim_A F = 1$ (failure of WSC).

for self-similar $F \subseteq \mathbb{R}$. (see García Adv. 2020 for $\mathbb{R}^d$ case)

Application: consider IFS in $[0,1]^2$ built by "lifting" the previous IFS.

projection π onto first co-ordinate is Lipschitz

$$(x,y) \xrightarrow{\pi} x$$



2D IFS satisfies OSC, and therefore $\dim_A E = s < 1$.

$$\alpha^s + \beta^s + \gamma^s = 1$$

1D IFS fails WSC $\dim_A F = 1$

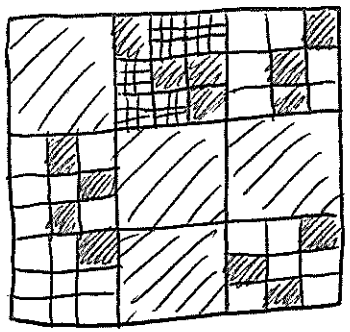
$$1 = \dim_A F = \dim_A \pi E > \dim_A E = s.$$

$\Rightarrow$ Lipschitz maps can increase Assouad dimension!

Mandelbrot percolation

Start with $[0,1]^d$, $m \geq 2$, $p \in (0,1)$

e.g. $d=2$ $m=3$



$m \times m$ grid

independently with prob. p keep

" " $(1-p)$ throw away.

$M_0 = [0,1]^d$

$M_k = \bigcup$ kept cubes at level k.

$M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ $\qquad M = \bigcap_k M_k$.

M is a random fractal. If p large enough, $\mathbb{P}(M \neq \emptyset) > 0$ and we can ask about almost sure dimensions of M conditioned on $M \neq \emptyset$.

<u>Theorem</u>

$$\dim_H M \overset{a.s}{=} \frac{\log(p\,m^d)}{\log m} \quad \left(= \frac{\log(\#\text{maps})}{-\log(\text{contraction})} \right)$$

<u>Theorem</u> Fraser - Miao - Troscheit (ETDS 2018)
& Berlinkov - Järvenpää (2019 JTP)

$$\dim_A M \overset{a.s}{=} d$$

Note: formula does not depend on m or p.

choosing $p > m^d$ but $p < m^d + \varepsilon$ we can ensure $\dim_H M$ almost surely < 0.0001, but $\dim_A M = d$.

Corollary

Almost surely M cannot be bi-Lipschitz embedded into $\mathbb{R}^{d-1}$!

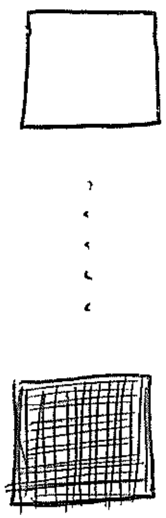
Proof: $\dim_A M \overset{a.s}{=} d$ and $\dim_A$ is bi-Lipschitz invariant.

Notes: • Assouad dimension has many nice applications in embedding theory.

• This gives a set with $\dim_H M < 0.0001$ and $M \subseteq \mathbb{R}^{100}$, such that M cannot be reasonably described in $\mathbb{R}^{99}$.

"Proof": $\dim_H F$ is big when F is "globally big".

$\dim_A F$ is big when F is big "somewhere".



Q kept at level k.

n more levels

cubes at level $k+n$.

"n experiment"

$$p = \mathbb{P}\left(\begin{array}{l} \text{all level } k+n \text{ cubes in } Q \text{ are kept} \\ \text{and go on to intersect } M \end{array} \right) > 0$$

- depends on n and p, but not on Q or k.

$l(n)$ times

Q k_1-level — n experiments

$k_1 + n$

Q k_2-level — n experiment

$\vdots$

Q $k_{l(n)}$-level. — n experiment

$\mathbb{P}(\text{success})$

$= \mathbb{P}(\text{success of one of experiments})$

$= 1 - (1-p)^{l(n)}$

$> 1/2$

for $l(n)$ large enough.

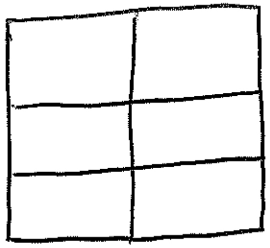
For each $n \in \mathbb{N}$, run $l(n)$ n-experiments.

Probability of success for each n is $> 1/2$.

Borel-Cantelli lemma $\Rightarrow$ infinitely many n are successful almost surely

$$\Rightarrow \dim_A M \overset{a.s.}{=} d.$$

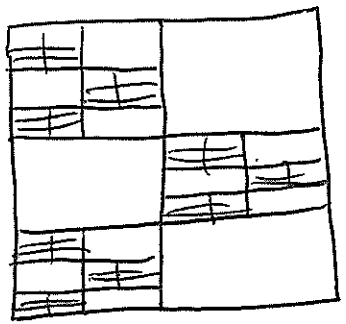
Self-affine Bedford–McMullen carpets



$[0,1]^2$, $n > m \geq 2$

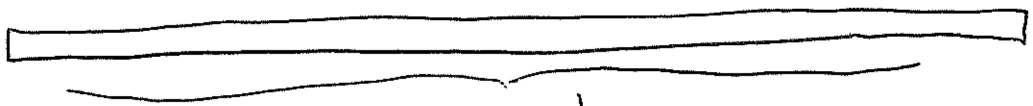
$m \times n$ grid
(e.g. $m=2$, $n=3$)

→ choose a sub-collection of rectangles and form an IFS (of affine maps)



F is the attractor.

basic cylinders are usually not good covers:



$\}\ n^{-k}$

m^{-k}

Theorem Bedford, McMullen 1984
Mackay 2011

N = total number of rectangles

M = total number of non-empty columns

N_i = number of rectangles in column i

$$N_{max} = \max_i N_i$$

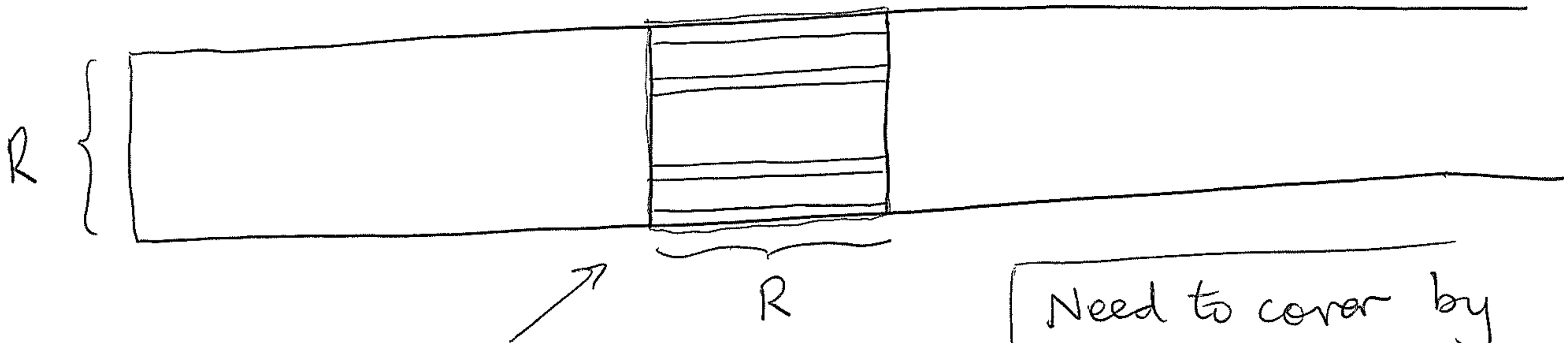
$$\dim_A F = \frac{\log M}{\log m} + \frac{\log N_{max}}{\log n}$$

$$\dim_B F = \frac{\log M}{\log m} + \frac{\log N/M}{\log m}, \quad \dim_H F = \frac{\log \sum_{i=1}^{M} N_i^{\log m / \log n}}{\log m}$$

Note: $\dim_H F < \dim_B F < \dim_A F$

provided $N_i = N_{max}$ for all i.
(non-uniform fibres case).

<u>Proof</u> For Assouad dimension, suffices to consider "approximate squares".



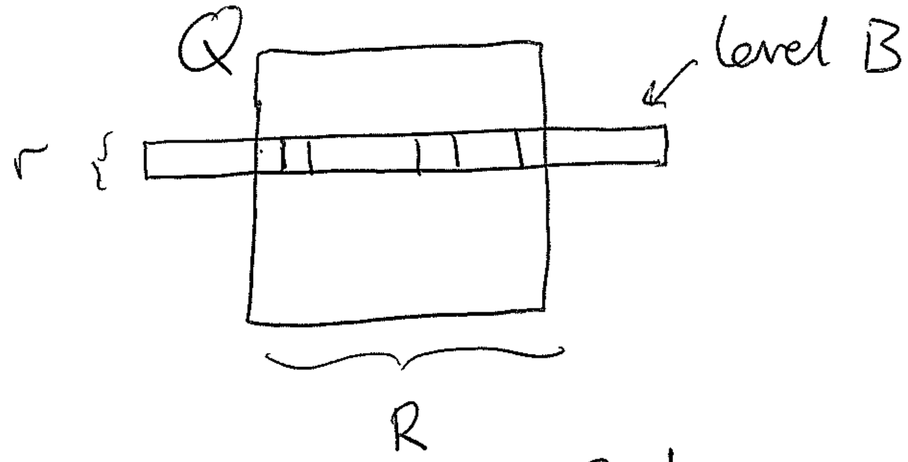
approximate square of size R.

Need to cover by squares of size r.

Define integers $a, A, b, B \geqslant 1$, by

$$n^{-b} \approx R \qquad m^{-a} \approx R \qquad \left(\begin{array}{ccc} b \leqslant & B & \leqslant \\ & & A \\ \leqslant & a \leqslant & \end{array} \right)$$

$$n^{-B} \approx r \qquad m^{-A} \approx r$$

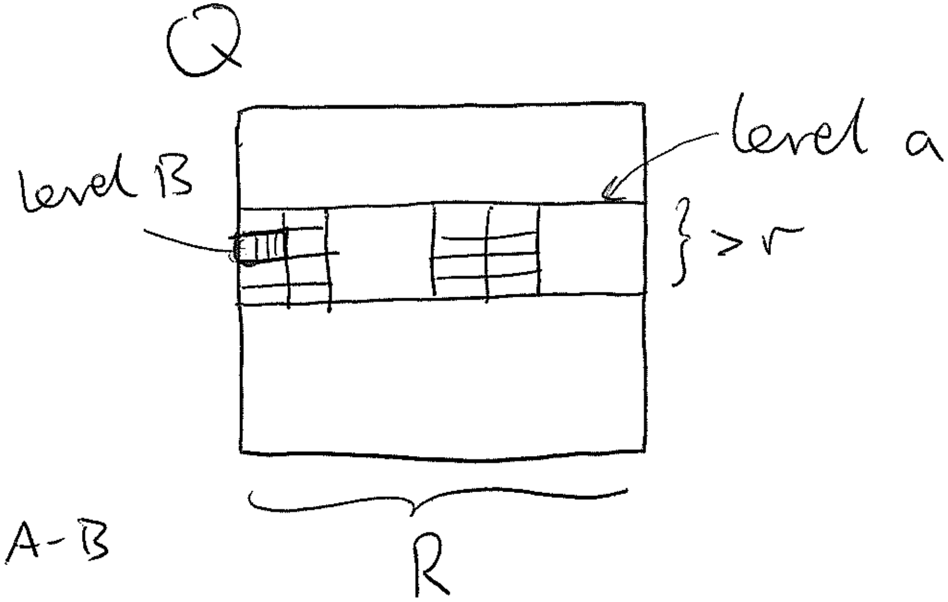


Case 1 $\quad b \leq B \leq a \leq A$

$$N_r(Q) \approx \left(\prod_{b}^{B} N_i \right) M^{A-a} \leqslant N_{max}^{B-b} M^{A-a}$$

$$\approx \left(\frac{R}{r} \right)^{\frac{\log M}{\log m} + \frac{\log N_{max}}{\log n}}$$

case 2 $\quad b \lesssim a \le B \le A$



$$N_r(Q) \approx \left(\prod_b^a N_i\right) N^{B-a} M^{A-B}$$

$$\lesssim N_{max}^{a-b} \left(M N_{max}\right)^{B-a} M^{A-B}$$

$$\boxed{N \le M N_{max}}$$

$$= N_{max}^{B-b} M^{A-a} \quad \text{(same as previous)}$$

$$\approx \left(\frac{R}{r}\right)^{\frac{\log M}{\log m} + \frac{\log N_{max}}{\log n}}$$

This proves upper bound, and lower bound now easy, (check case 1 and consider only place where non optimal estimate used)

Assouad dimension & Fractal Geom.

Exercises (part 2)

(14) Prove directly from the definition that $\dim_A F = s$ for all self-similar sets satisfying OSC (or SSC) where s is similarity dimension ($\sum_i c_i^s = 1$).

(15) Prove directly that if an IFS on $[0,1]$ contains two similarities with a common fixed point and contraction ratios α, β with $\frac{\log \alpha}{\log \beta} \notin \mathbb{Q}$, then

$\dim_A F = 1$ (where F is attractor)

(16) Construct a self-similar set with distinct Hausdorff and Assouad dimensions, but using only one contraction ratio.

(17) Construct an infinitely generated self-similar set ($|\mathcal{I}| = \infty$) which satisfies the SSC (or OSC) but has distinct Hausdorff and Assouad dimensions.

(18) Consider a more general percolation model where m and p are allowed to change at each level. What can you say about Assouad dimension?

(19) Construct a self-similar set F in $\mathbb{R}^2$ which satisfies:

$\dim_H F < \dim_A F < 2$.

(20) Find out what an Ahlfors regular set is and show that for such sets F, $\dim_A F = \dim_H F$.